

Properties of random nilpotent groups

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Abstract

We study random nilpotent groups of the form $G = N/\langle\langle R \rangle\rangle$, where N is a non-abelian free nilpotent group with m generators, and R is a set of r random relators of length ℓ . We prove that the following holds asymptotically almost surely as $\ell \rightarrow \infty$: 1) If $r \leq m - 2$, then the ring of integers \mathbb{Z} is e-definable in $G/Is(G_3)$, and systems of equations over \mathbb{Z} are reducible to systems of equations over G (hence, they are undecidable). Moreover, $Z(G) \leq Is(G')$, G/G_3 is virtually free nilpotent, and G/G_3 cannot be decomposed as the direct product of two non-virtually abelian groups. 2) If $r = m - 1$, then G is virtually abelian. 3) If $r = m$, then G is finite. 4) If $r \geq m + 1$, then G is finite and abelian. In the last three cases, systems of equations are decidable in G .

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1 Introduction

In [1], Cordes, Duchin, Duong, Ho, and Sánchez introduced a model of random finitely generated nilpotent groups. Such model is the analog of the *few-relators* and the *density* models of random finitely presented groups, where one takes a free group $F_m = F_m(a_1, \dots, a_m)$, and then adds a set of random relations R . Every relator is chosen among all words of a certain length ℓ on the alphabet $A^{\pm 1} = \{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$, with uniform probability. The length ℓ is thought of as an integer variable that tends to infinity, and the number of chosen relators is taken to be a function of ℓ . For instance, $|R| = (2m + 1)^{d\ell}$ ($0 < d < 1$) in the density model, whereas $|R|$ is constant in the few-relators model. One can then consider the probability p_ℓ that a group $G = F_m / \langle\langle R \rangle\rangle$ satisfies some property P , for a fixed ℓ . The limit $p = \lim_{\ell \rightarrow \infty} p_\ell$, if it exists, is called the *asymptotic probability* that G satisfies P . If $p = 1$, then G is said to satisfy P *asymptotically almost surely* (a.a.s.) For example, a well-known result of Gromov [5] states that, in the density model, G is hyperbolic if $d < 1/2$, and finite if $d > 1/2$, a.a.s. See [10] for more information on random f.p. groups.

Since all finitely generated nilpotent groups are quotients of free nilpotent groups by some finite set of relators, one can easily adapt this procedure to the class of f.g. nilpotent groups: it suffices to replace F_m by an s -step rank- m free nilpotent group $N_{s,m} = N_{s,m}(A)$. Then, as before, one chooses a set R of random words of length ℓ on the alphabet $A^{\pm 1}$. These words are added as relators to $N_{s,m}$, yielding a random f.g. nilpotent group $G = N_{s,m} / \langle\langle R \rangle\rangle$. In [1] it is proved (among other results) that G is trivial a.a.s. (as $\ell \rightarrow \infty$) if and only if $|R|$ tends to infinity as a function of ℓ . In particular, the analog of the density model yields trivial groups a.a.s. for any density parameter d . For this reason, in the nilpotent case the few-relators model in which the cardinality $|R|$ is constant, seems more appropriate. Notice that the approach above is based on the fact that the class of s -step nilpotent groups forms a variety (defined by the identity $[x_1, \dots, x_{s+1}] = 1$), so there are free objects there (free nilpotent groups) and every finitely generated group in the variety is a quotient of a finitely generated free nilpotent group. The same approach can be used to define random groups in other varieties.

Note that there is another model that gives random 2-generated torsion-free nilpotent groups: in [2] such groups are obtained as subgroups of groups of unitriangular matrices generated by two random words in the standard generators. This model exploits the classical result that any finitely generated torsion-free nilpotent group embeds into a suitable unitriangular group $UT_n(\mathbb{Z})$.

In this paper, following [1], we study the structure of random nilpotent groups. As a corollary we characterize decidability of Diophantine problem in random nilpotent groups. Below we fix some terminology:

Definition 1.1. The *Diophantine problem* over an algebraic structure \mathcal{A} , denoted $\mathcal{D}(\mathcal{A})$, is a task to determine whether or not a given finite system of equations over \mathcal{A} (\mathcal{A} -systems) has a solution in \mathcal{A} . $\mathcal{D}(\mathcal{A})$ is *decidable* if there is an algorithm which for a given finite system S of equations in \mathcal{A} decides

whether S has a solution in \mathcal{A} or not, otherwise, $\mathcal{D}(\mathcal{A})$ is *undecidable*. Furthermore, $\mathcal{D}(\mathcal{A})$ is *m-reducible* to $\mathcal{D}(\mathcal{M})$ for another structure \mathcal{M} , if there is an algorithm that for any finite system of equations S over \mathcal{A} computes a finite system of equations $S_{\mathcal{M}}$ over \mathcal{M} such that S has a solution in \mathcal{A} if and only if $S_{\mathcal{M}}$ has a solution in \mathcal{M} .

In this paper, a reduction always means an m-reduction, so we omit m from the notation. Notice that due to the classical result of Davis, Putnam, Robinson and Matiyasevich Diophantine problem $\mathcal{D}(\mathbb{Z})$ is undecidable. Hence if $\mathcal{D}(\mathbb{Z})$ is reducible to $\mathcal{D}(\mathcal{M})$, then $\mathcal{D}(\mathcal{M})$ is also undecidable. As far as we know this idea was used in all the results on undecidability of equations in nilpotent groups, see [11, 3, 4] and the surveys of results in the last two papers.

The following is the main result of the paper. To explain our results we need some notation. Below, G_i is the i -th term of the lower central series of G defined inductively by $G_1 = G, G_{i+1} = [G_i, G]$. As usual, we denote G_2 also by G' . $Is(G_i)$, the *isolator* of G_i , is defined as $Is(G_i) = \{g \in G \mid g^n \in G_i \text{ for some } n \in \mathbb{Z}, n \neq 0\}$. Recall [8] that a nilpotent group G is called *regular* if $Z(G) \leq Is(G')$.

Below, given $R = \{w_1, \dots, w_r\}$, we denote by $M(R)$ the $r \times m$ matrix whose (i, j) -th entry is the sum of all the exponents of the a_j 's appearing in w_i .

Theorem 1.2. *Let $G = N_{s,m}/\langle\langle R \rangle\rangle$, where $N_{s,m} = N_{s,m}(A)$ is a free nilpotent group of nilpotency class $s \geq 2$ and rank m with basis A and $R = \{w_1, \dots, w_r\}$ is a set of r words of length ℓ in the alphabet $A^{\pm 1}$. Then the matrix $M(R)$ has full rank a.a.s., which implies that:*

1. *If $r \leq m - 2$, then $Z(G) \leq Is(G')$, G/G_3 is virtually free nilpotent of rank $m - r$, and G/G_3 does not decompose as the direct product of two non-virtually abelian groups.*
2. *If $r = m - 1$, then G is virtually abelian.*
3. *If $r = m$, then G is finite.*
4. *If $r \geq m + 1$, then G is finite and abelian.*

The statement concerning abelianity in Item 4 is proved by Cordes, Duchin, Duong, Ho, and Sánchez in [1]. Among other results, they also show that G may be abelian in the cases $r = m - 1$ and $r = m$ (with nonzero asymptotic probability).

The proof of Theorem 1.2 is divided into two parts. In the first one we forget about random nilpotent groups and we show the following:

Theorem 3.7. *Let G be a nilpotent group of the form $G = N_{s,m}(A)/\langle\langle R \rangle\rangle$, for R a set of words on the alphabet $A^{\pm 1}$. Suppose that $M(R)$ has full rank, i.e. $\text{rank}(M(R)) = \min\{r, m\}$. Then the conclusions of Theorem 1.2 hold for the group G .*

In the second part we show that if the words from R are chosen randomly, then $M(R)$ has full rank a.a.s.:

Theorem 4.1. *Let R be a set of r words, each one chosen randomly among all words of length ℓ on the alphabet $A^{\pm 1} = \{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$. Then $M(R)$ has full rank (i.e. $\text{rank}(M(R)) = \min\{r, m\}$) asymptotically almost surely as $\ell \rightarrow \infty$.*

Theorem 1.2 now follows from Theorems 3.7 and 4.1. We also study the Diophantine problem over random nilpotent groups, showing that:

Theorem 1.3. *Let $G = N_{s,m}/\langle\langle R \rangle\rangle$, where $N_{s,m} = N_{s,m}(A)$ is a free nilpotent group of nilpotency class $s \geq 2$ and rank m with basis A and R is a set of r words of length ℓ in the alphabet $A^{\pm 1}$. Then the following hold asymptotically almost surely as $\ell \rightarrow \infty$:*

1. *If $r \leq m - 2$, then \mathbb{Z} is e -interpretable in G and the Diophantine problem over G is undecidable.*
2. *If $r \geq m - 1$ then Diophantine problem over G is decidable.*

Again, this result is proved by using Theorem 4.1 and showing that its conclusions hold when $\text{rank}(M(R)) = \min\{r, m\}$ (see Theorem 3.8).

2 Preliminaries

2.1 Nilpotent groups

Following standard conventions, we call the element $[g, h] = g^{-1}h^{-1}gh$ of a group G the *commutator* of g and h , and we denote the subgroup of G formed by all its commutators by G' . More generally, we define inductively $G_1 = G$, $G_2 = G' = \langle [G, G] \rangle = \langle \{[g, h] \mid g, h \in G\} \rangle$, and $G_{n+1} = \langle [G, G_n] \rangle$. The subnormal series $G_1 \supseteq G_2 \supseteq \dots$ is called the *lower central series* of G . If $G_{s+1} = 1$ for some $s \geq 1$, then G is said to be *s-step nilpotent*, or just *nilpotent*. For example, G is 1-step nilpotent if and only if it is an abelian group, and it is 2-step nilpotent if and only if

$$G_3 = \langle [G, [G, G]] \rangle = \langle \{[g_1, [g_2, g_3]] \mid g_i \in G\} \rangle = 1,$$

which is the same as saying that all commutators belong to the center of G , i.e. to $Z(G) = \{g \in G \mid [g, h] = 1 \text{ for all } h \in G\}$. The element $[g_1, [g_2, g_3]]$ is called a 3-fold commutator, and it is usually denoted by $[g_1, g_2, g_3]$. Inductively, an *n-fold commutator* is defined as $[g_1, \dots, g_n] = [g_1, [g_2, \dots, g_n]]$.

The identities $[ab, c] = b^{-1}[a, c]b[b, c]$ and $[a, bc] = [a, c]c^{-1}[a, b]c$ hold in any group G . In particular, if G is 2-step nilpotent, then:

$$[ab, c] = [a, c][b, c] \quad \text{and} \quad [a, bc] = [a, b][a, c] \quad \text{for all } a, b, c \in G.$$

In this case, informally speaking, $[,]$ can be thought of a ‘bilinear map’ $[,] : G \times G \rightarrow G$. We will use this fact implicitly from now on.

Let $F_m = F_m(A)$ be the free group of rank m with generating set $A = \{a_1, \dots, a_m\}$, and let $T_{j,m} = \{[a_{i_1}, \dots, a_{i_j}] \mid 1 \leq i_1, \dots, i_j \leq m\}$ be the set of

all j -fold commutators on the a_i 's. Then the *free s -step rank- m nilpotent group with basic generating set A* is

$$N_{s,m} = N_{s,m}(A) = F_m / \langle\langle T_{s+1,m} \rangle\rangle = \\ = \langle a_1, \dots, a_m \mid [a_{i_1}, \dots, a_{i_{s+1}}] = 1 \text{ for all } i_j \rangle.$$

Here $\langle\langle T_{j,m} \rangle\rangle$ denotes the normal closure of $T_{j,m}$ in F_m . For $s = 2$, the set $\mathcal{A} = A \cup \{[a_i, a_j] \mid i < j\}$ is a so-called *Malcev basis* of $N_{2,m}(A)$. Any $g \in N_{2,m}(A)$ admits the expression

$$g = a_1^{\alpha_1} \dots a_m^{\alpha_m} \prod_{1 \leq i < j \leq m} [a_i, a_j]^{\gamma_{i,j}}, \quad (\alpha_k, \gamma_{i,j} \in \mathbb{Z}), \quad (1)$$

which is *unique* up to the order of the commutators $[a_i, a_j]$. This follows from the fact that $\{[a_i, a_j] \mid i < j\}$ forms a basis of N' , while A projects onto a basis of N/N' . Usually, (1) is called the *Malcev representation* of g (with respect to \mathcal{A}), and the integers $\alpha_k, \gamma_{i,j}$ are called the *Malcev coordinates* of g .

We will need the following result:

Lemma 2.1. *Let G be a finitely generated nilpotent group. Assume G' is finite. Then G is virtually abelian.*

Proof. We now prove the second item using induction on s . The case $s = 1$ is clear. Suppose that $s > 1$ and that G is non-abelian. Let $\pi : G \rightarrow G/G_s$ denote the natural projection of G onto G/G_s . Notice that $\pi(G)$ is $(s-1)$ -step nilpotent, and that $\pi(G)' = \pi(G')$ is finite, because G' is. By induction, $\pi(G)$ is virtually abelian, and hence it has an abelian subgroup of finite index. Let H be the full pre-image in G by π of such subgroup. We have $\ker(\pi) \leq H \leq G$, and therefore

$$|G : H| = |G/\ker(\pi) : H/\ker(\pi)| < \infty.$$

Moreover, since $\pi(H)$ is abelian, $\pi(H') = \pi(H)' = 1$, and thus $H' \leq \ker(\pi) = G_s$. It follows that

$$[H, H'] \subseteq [H, G_s] \subseteq [G, G_s] \subseteq G_{s+1} = 1.$$

Since $H_3 = \langle [H, H'] \rangle$, we obtain that H is 2-step nilpotent. Note that H' is finite because $H' \leq G_s \leq G'$, and G' is finite by hypothesis. Let t denote the cardinality of H' . Then

$$[h_1, h_2^t] = [h_1, h_2]^t = 1$$

for any two $h_1, h_2 \in H$. Hence, $H^t = \{h^t \mid h \in H\}$ belongs to the center of H . Consequently, $\langle H^t \rangle$ is an abelian normal subgroup of H , and H/H^t is a finitely generated nilpotent group of finite exponent t . By [14], $H/\langle H^t \rangle$ is finite. We conclude that $\langle H^t \rangle$ is a finite-index abelian subgroup of H . This means that H is virtually abelian. Since H has finite index in G , so is G . \square

2.2 E-definability

In what follows we often use non-cursive boldface letters to denote tuples of elements: e.g. $\mathbf{a} = (a_1, \dots, a_n)$.

Definition 2.2. Let $\mathcal{M} = (M; f_i, r_j, c_k \mid i, j, k)$ be an algebraic structure (for the purposes of this paper, \mathcal{M} is a group or a ring), where M is the universe set of \mathcal{M} , and f_i, r_j, c_k are the function, relation, and constant symbols of \mathcal{M} . A set $A \subseteq M^m$ is called *definable by equations* in \mathcal{M} , or *e-definable*, if there exists a finite system of equations over \mathcal{M} , $\Sigma_A(x_1, \dots, x_m, y_1, \dots, y_n)$, on variables $\mathbf{x} = (x_1, \dots, x_m) \in M^m$ and $\mathbf{y} = (y_1, \dots, y_n) \in M^n$, such that, for any tuple $\mathbf{a} \in M^m$, we have that $\mathbf{a} \in A$ if and only if $\Sigma_A(\mathbf{a}, \mathbf{y})$ has a solution $\mathbf{y} \in M^n$.

A function $f : X_1 \times \dots \times X_k \subseteq M^k \rightarrow M^l$ is called *e-definable* in \mathcal{M} if its graph $\{(a_1, \dots, a_k, f(a_1, \dots, a_k)) \mid a_i \in X_i\} \subseteq M^{k+l}$ is e-definable in \mathcal{M} . Similarly, a relation $r : X_1 \times \dots \times X_k \rightarrow \{0, 1\}$ is *e-definable* in \mathcal{M} if its graph $\{(a_1, \dots, a_k) \mid r(a_1, \dots, a_k) = 1\}$ is e-definable in \mathcal{M} .

Definition 2.3. An algebraic structure $\mathcal{A} = (A; f, \dots, r, \dots, c, \dots)$ is called *e-definable* in another structure $\mathcal{M} = (M; \dots)$ if there exists an injective map $\phi : A \hookrightarrow M^k$ for some k , called *defining map*, such that:

1. The image of ϕ is e-definable in \mathcal{M} .
2. For every function $f = f(x_1, \dots, x_n)$ of \mathcal{A} , the induced function $\phi(f)$ defined as $\phi(f)(\phi(x_1), \dots, \phi(x_n)) = \phi(f(x_1, \dots, x_n))$ is e-definable in \mathcal{M} .
3. Similarly, for every relation r of \mathcal{A} , the induced relation $\phi(r)$ (with a meaning analogous to that of $\phi(f)$) is e-definable in \mathcal{M} .

For e-definability we always assume that $=$ is a relation of both \mathcal{A} and \mathcal{M} .

Example 2.4. The center $Z(G)$ of a finitely generated group $G = \langle g_1, \dots, g_n \rangle$ is e-definable in G as a set. Indeed, $x \in G$ belongs to $Z(G)$ if and only if it commutes with all g_i 's, and hence $Z(G)$ (seen as a set) is defined in G by means of the following system of equations on the single variable x :

$$\bigwedge_{i=1}^n ([x, g_i] = x^{-1} g_i^{-1} x g_i = 1).$$

If, additionally, we regard $Z(G) = (Z(G); \cdot, ^{-1}, 1)$ as an algebraic structure with operations and constants inherited from G , then $Z(G)$ is still e-definable in G with defining map $id : Z(G) \rightarrow G$, $id(g) = g$.

For another example, let G be a group with finite $[x, y]$ -width n (see below in this section). Then any $g \in G'$ can be written as a product of exactly n commutators (adding trivial ones if necessary), and thus G' is e-definable in G by means of the equation $x = [x_1, y_2] \cdots [x_n, y_n]$.

Further examples can be found in Proposition 2.8, and in the *verbal width* subsection below.

Given a tuple $\mathbf{a} = (a_1, \dots, a_n) \in A^n$ and a map $\phi : A \rightarrow M^k$, we denote by $\phi(\mathbf{a})$ the tuple in M^{kn} consisting in the components of $\phi(a_1)$, followed by the components of $\phi(a_2)$, and so on. The following is a fundamental property of e-definability:

Lemma 2.5. *If \mathcal{A} is e-definable in \mathcal{M} (with defining map $\phi : A \rightarrow M^k$), then for every system of equations $S(\mathbf{x}) = S(x_1, \dots, x_n)$ over \mathcal{A} , there exists a system of equations $S^*(\mathbf{y}, \mathbf{z}) = S^*(y_1, \dots, y_{kn}, z_1, \dots, z_m)$ over \mathcal{M} , such that \mathbf{a} is a solution to S in \mathcal{A} if and only if $S^*(\phi(\mathbf{a}), \mathbf{z})$ has a solution \mathbf{z} in \mathcal{M} . Moreover, all solutions \mathbf{b}, \mathbf{c} to S^* arise in this way, i.e. $\mathbf{b} = \phi(\mathbf{a})$ for some solution \mathbf{a} to S .*

Proof. It suffices to follow step by step the proof of Theorem 5.3.2 from [6], which states that the above holds when \mathcal{A} is interpretable by first order formulas in \mathcal{M} . \square

Following the notation of Lemma 2.5, S has a solution in \mathcal{A} if and only if S^* has a solution in \mathcal{M} . Recall that $\mathcal{D}(\mathcal{A})$ denotes the Diophantine problem over \mathcal{A} (see Definition 1.1). One immediately obtains:

Corollary 2.6. *If \mathcal{A} is e-definable in \mathcal{M} , then $\mathcal{D}(\mathcal{A})$ is reducible to $\mathcal{D}(\mathcal{M})$. Consequently, if $\mathcal{D}(\mathcal{A})$ is undecidable, then $\mathcal{D}(\mathcal{M})$ is undecidable as well.*

The following is an application of the so-called notion of e-interpretability, a generalization of e-definability that avoids the restrictive requirement of A embedding into M^n .

Proposition 2.7. *Let N, H, G be three groups such that $N \leq H \leq G$, N is normal in G , and both N and H are e-definable in G . Then $\mathcal{D}(H/N)$ is reducible to $\mathcal{D}(G)$.*

Proof. Any single equation $w(\mathbf{z})N = N$ over G/N on n variables \mathbf{z} is equivalent to the problem of finding n elements $\mathbf{g} \in G^n$ such that $w(\mathbf{g}) \in N$. Since N is e-definable in G , there exists a system $S_w = S_w(\mathbf{x}_w, \mathbf{y}_w)$ over G such that $w(\mathbf{g}) \in N$ if and only if $S_w(\mathbf{g}, \mathbf{y}_w)$ has a solution \mathbf{y}_w in G . Of course, \mathbf{x}_w is a tuple of n variables, each one taking values in G . Moreover, since H is e-definable in G , there exists a system $S_H(z, \mathbf{v})$ such that an element $g \in G$ belongs to H if and only if $S_H(g, \mathbf{v})$ has a solution in \mathbf{v} .

Now, given a system $S = \{w_i(\mathbf{z})N = N \mid i\}$ over H/N , on n variables \mathbf{z} , let S^* be the system over G formed by all the $S_{w_i}(\mathbf{x}_{w_i}, \mathbf{y}_{w_i})$'s, the equations $\mathbf{x}_{w_1} = \dots = \mathbf{x}_{w_i} = \dots$, and systems $\{S_H(x_{ij}, \mathbf{v}_{ij}) \mid i, j\}$ that ensure that each variable x_{ij} in each tuple \mathbf{x}_{w_i} takes values in H . Then S has a solution in H/N if and only if S^* has a solution in G . This gives a reduction of $\mathcal{D}(H/N)$ to $\mathcal{D}(G)$. \square

Similar arguments as above can be used to show the stronger statement that G/N is e-interpretible in G . We next use Proposition 2.7 to study the torsion subgroup of a nilpotent group, "equation-wise".

Proposition 2.8. *Let T be the torsion subgroup of a finitely generated nilpotent group G . Then T is e-definable in G , and, consequently, $\mathcal{D}(G/T)$ is reducible to $\mathcal{D}(G)$.*

Proof. The torsion subgroup of such G has finite order n ([14], Chapter 1, Corollary 10). Hence, the equation $x^n = 1$ e-defines T in G . The result now follows from Proposition 2.7. \square

2.3 Verbal width

Let $w = w(x_1, \dots, x_m)$ be a word on an alphabet of variables $\{x_1, \dots, x_m\}$. The *w-verbal subgroup* of a group G is defined as $w(G) = \langle w(g_1, \dots, g_m) \mid g_i \in G \rangle$, and G is said to have *finite w-width* if there exists an integer n , such that every $g \in w(G)$ can be expressed as a product of at most n elements of the form $w(g_1, \dots, g_m)^{\pm 1}$, i.e., if, for all $g \in w(G)$,

$$g = \prod_{i=1}^{n'} w(g_1^i, \dots, g_m^i)^{\epsilon_i} \text{ for some } g_j^i \in G, \epsilon_i \in \{-1, +1\}, \text{ and } n' \leq n.$$

In this case, each $g \in w(G)$ can be expressed as a product of *exactly* n elements of the form $w(g_1, \dots, g_m)^{\pm 1}$ (adding trivial ones $w(1, \dots, 1)$ if necessary). Hence, $w(G)$ is e-definable in G through the equation

$$x = \prod_{i=1}^n w(y_{i1}, \dots, y_{im}) \prod_{i=1}^n w(z_{i1}, \dots, z_{im})^{-1}.$$

on variables x and $\{y_{ij}, z_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$.

For example, if G is a finitely generated nilpotent group, then G_i (the i -th subgroup of the lower central series of G) is w_i -verbal and it has finite w_i -verbal width, for $w_i = [x_1, \dots, x_i]$, [12] [13]. In particular, G_i is e-definable in G .

Proposition 2.9. *Let G be a finitely generated nilpotent group with lower central series $G = G_1 \supseteq G_2 \supseteq \dots \supseteq G_s \supseteq G_{s+1} = 1$. Denote by H_{ij} the quotient group G_i/G_j modulo its torsion, for $1 \leq i \leq j \leq s+1$. Then $\mathcal{D}(H_{ij})$ is reducible to $\mathcal{D}(G)$, for all $i \leq j$.*

Proof. Since each G_k is e-definable and normal in G [12] [13], $\mathcal{D}(G_i/G_j)$ is reducible to $\mathcal{D}(G)$ for all $i \leq j$, by Proposition 2.7. Furthermore, G_i/G_j is a finitely generated nilpotent group, and hence, by Proposition 2.8, $\mathcal{D}(H_{ij})$ is reducible to $\mathcal{D}(G_i/G_j)$. Since reducibility is transitive, $\mathcal{D}(H_{ij})$ is reducible to $\mathcal{D}(G)$. \square

This proposition allows one to reduce $\mathcal{D}(H_{1,3})$ to $\mathcal{D}(G)$. This is of particular interest because $H_{1,3}$ is a finitely generated torsion-free 2-step nilpotent group, possibly non-abelian. There are other H_{ij} 's with these characteristics, for example $H_{i,3i}$, for any i .

3 Systems of equations over nilpotent groups

3.1 Small centralizers and e-definability of \mathbb{Z}

In this section we study how one can e-define the ring \mathbb{Z} in a finitely generated torsion-free 2-step nilpotent group. We start with a key concept:

Definition 3.1. An element g of a group G is said to be *centralizer-small*, or just *c-small*, if $C_G(g) = \{g^t z \mid t \in \mathbb{Z}, z \in Z(G)\}$, where $C_G(g)$ denotes the set of elements in G that commute with g .

The techniques used to prove the following result have a resemblance with some arguments from Duchin, Liang, and Shapiro in [1], and from Romankov in [11].

Theorem 3.2. *Let G be a finitely generated torsion-free 2-step nilpotent group, and suppose G has two non-commuting c-small elements a, b . Then \mathbb{Z} is e-definable in G . Consequently, $\mathcal{D}(\mathbb{Z})$ is reducible to $\mathcal{D}(G)$, and $\mathcal{D}(G)$ is undecidable.*

Proof. Consider the set $Z = [a, C_G(b)] = \{[a, x] \mid x \in C_G(b)\}$, and let $c = [a, b] \neq 1$. Using that b is c-small, $Z = \{c^t \mid t \in \mathbb{Z}\}$. Since G is torsion free, each $g = c^t \in Z$ may be identified with t . Define binary and unary operations \oplus, \ominus , in Z , by letting

$$c^{t_1} \oplus c^{t_2} = c^{t_1+t_2}, \quad \ominus(c^t) = c^{-t}, \quad \text{and} \quad c^{t_1} \odot c^{t_2} = c^{t_1 t_2}.$$

An element $g \in G$ belongs to Z if and only if the following identities hold for some $y \in G$: $g = [a, y]$, $[y, b] = 1$. In other words, $g \in Z$ if and only if g is part of a solution to the system of equations $(x = [a, y]) \wedge ([y, b] = 1)$ on variables x, y . Hence, Z is e-definable in G as a set. Now let $g_1, g_2, g_3 \in Z$. Clearly, $g_1 \oplus g_2 = g_3$ if and only if $g_1 g_2 = g_3$. It follows that the graph of \oplus is e-definable in G : it suffices to take the system formed by the equation $xy = z$ together with equations that ensure $x, y, z \in Z$. Analogously, and taking the equation $xy = 1$ instead of $xy = z$, one sees that \ominus is e-definable in G .

Regarding \odot , consider the following system over G on variables $x_i, i = 1, 2, 3$, and $x'_i, i = 1, 2$.

$$\begin{cases} x_1 = [x'_1, b], & [x'_1, a] = 1, \\ x_2 = [a, x'_2], & [x'_2, b] = 1, \\ x_3 = [x'_1, x'_2]. \end{cases} \quad (2)$$

Suppose $x_1, x_2, x_3, x'_1, x'_2$ is a solution to (2). Since a and b are c-small, $x'_1 = a^{t_1} z_1$ and $x'_2 = b^{t_2} z_2$ for some $t_i \in \mathbb{Z}$ and some $z_i \in Z(G)$, $i = 1, 2$. Moreover, $x_1 = c^{t_1}$ and $x_2 = c^{t_2}$. We also have $[x'_1, x'_2] = c^{t_1 t_2} = x_3$, and hence $x_3 = c^{t_1 t_2} = x_1 \odot x_2$. Conversely, let x_1, x_2, x_3 be three elements from G such that $x_1 \odot x_2 = x_3$. Then it is easy to verify that there exist x'_1, x'_2 such that $x_1, x_2, x_3, x'_1, x'_2$ form a solution to (2). We conclude that $x_1 \odot x_2 = x_3$ if and

only if x_1, x_2, x_3 are part of a solution to (2). Similarly as before, \odot is e-definable in G .

This completes the proof, since the ring $(\mathbb{Z}; \oplus, \ominus, \odot, c^0, c^1)$ is e-definable in G and it is isomorphic to the ring of integers $(\mathbb{Z}; +, -, \cdot, 0, 1)$. \square

Remark 3.3. One may extend Theorem 3.2 to the class of finitely generated nilpotent groups G (of any nilpotency step, possibly with torsion): by Proposition 2.9, it suffices to take (if they exist) two non-commuting c-small elements in any quotient group $H_{i,j}$ with $j > 2i$. For example, in the next section we will study $H_{1,3}$.

We recover one of the results from [3]:

Corollary 3.4. *\mathbb{Z} is e-interpretable in any non-abelian free nilpotent group N . Consequently, $\mathcal{D}(\mathbb{Z})$ is reducible to $\mathcal{D}(N)$, and $\mathcal{D}(N)$ is undecidable.*

Proof. Suppose first that N is a non-abelian 2-step free nilpotent with basic generating set $A = \{a_1, \dots, a_n\}$. Let $C = \{[a_i, a_j] \mid 1 \leq i < j \leq n\}$. Using that $(A; C)$ is a Malcev basis of N , and bilinearity of $[\cdot, \cdot]$, one sees that a_1 and a_2 are two non-commuting c-small elements of N (this is done in for example in [3], without the c-small terminology). Since N is a τ_2 -group, \mathbb{Z} is e-definable in N , by Theorem 3.2.

Now assume that N is s -step nilpotent for some $s > 2$, and let N_3 be the third term of its lower central series. Then N/N_3 is a non-abelian 2-step free nilpotent group, and by the previous paragraph, \mathbb{Z} is e-definable in N/N_3 . By Proposition 2.9, N/N_3 is e-interpretable in N . Finally, since e-interpretability is a transitive property, \mathbb{Z} is e-interpretable in N . Hence, by Corollary 2.6, $\mathcal{D}(\mathbb{Z})$ is reducible to $\mathcal{D}(N)$. This makes $\mathcal{D}(N)$ undecidable. \square

3.2 Rings of scalars and direct decompositions

A ring of scalars of a τ_2 -group G is a commutative, associative ring R with identity such that R acts faithfully by endomorphisms on $G/Z(G)$ and on G' , and such that the commutator map between abelian groups

$$\begin{aligned} [\cdot, \cdot] : G/Z(G) \times G/Z(G) &\rightarrow G' \\ (gZ(G), hZ(G)) &\mapsto [g, h] \end{aligned}$$

is R -bilinear. R is called *largest* if $K \leq R$ for any other ring of scalars of G . It is possible to define ring of scalars for arbitrary nilpotent groups (and not just τ_2) [9, 8]. Note that, due to the choice of notation, $[g, h] = [gZ(G), hZ(G)]$ for all $g, h \in G$.

A τ_2 -group will refer to a finitely generated torsion-free 2-step nilpotent group.

Theorem 3.5. *Let a_1, \dots, a_n be a set of generators of a τ_2 -group G ($n \geq 2$). Assume that $[a_i, a_j] \neq 1$ for all $i \neq j$, and that a_i is c-small for all $i = 1, \dots, n$. Then the largest ring of scalars of G is isomorphic to the ring of integers \mathbb{Z} .*

Proof. Let $\bar{\cdot}$ denote the natural projection from G to $G/Z(G)$. Suppose R is a ring of scalars of G , and fix an element $r \in R$. For each $i = 1, \dots, n$, choose a representative $b_{r,i}$ of $r\bar{a}_i$, so that $\bar{b}_{r,i} = r\bar{a}_i$. Then

$$[a_i, b_{r,i}] = [\bar{a}_i, \bar{b}_{r,i}] = [\bar{a}_i, r\bar{a}_i] = r[\bar{a}_i, \bar{a}_i] = r[a_i, a_i] = 1,$$

and hence a_i and $b_{r,i}$ commute for all i . Since a_i is c -small, $b_{r,i} = a_i^{t_{r,i}} z_i$ for some $t_{r,i} \in \mathbb{Z}$ and some $z_i \in Z(G)$. Thus, $r\bar{a}_i = \bar{a}_i^{t_{r,i}}$. Now, for any i, j :

$$\begin{aligned} [a_i, a_j]^{t_{r,i}} &= [a_i^{t_{r,i}}, a_j] = [r\bar{a}_i, \bar{a}_j] = \\ &= [\bar{a}_i, r\bar{a}_j] = [\bar{a}_i, \bar{a}_j^{t_{r,j}}] = [a_i, a_j^{t_{r,j}}] = [a_i, a_j]^{t_{r,j}}. \end{aligned}$$

Since G is torsion-free and $[a_i, a_j] \neq 1$, we obtain $t_{r,i} = t_{r,j}$ for all $i \neq j$. It follows that, for all $r \in R$, there exists $t_r \in \mathbb{Z}$ such that $r(\bar{a}_i) = \bar{a}_i^{t_r}$ for all i .

Each \bar{g} can be written as $\bar{g} = \prod_i \bar{a}_i^{k_i}$, and since R acts by endomorphisms on $G/Z(G)$,

$$r\bar{g} = \prod_i r\bar{a}_i^{k_i} = \prod_i \bar{a}_i^{t_r k_i} = \left(\prod_i \bar{a}_i^{k_i} \right)^{t_r} = \bar{g}^{t_r}.$$

In conclusion, $r\bar{g} = \bar{g}^{t_r}$ for all $g \in G$.

Let $\phi : R \rightarrow \mathbb{Z}$ be the map $\phi(r) = t_r$. We next show that ϕ is a ring homomorphism. To clarify our arguments we will write \oplus , \odot , 1_R and 0_R when referring to addition and multiplication in R , and to its identity and zero element, respectively. Recall though that R is identified with its embedding into $\text{End}(G/Z(G))$. Hence \oplus is addition of endomorphisms, \odot is composition, 1_R is the identity endomorphism, and 0_R is the endomorphism that sends all elements to the identity element of $G/Z(G)$. For all other structures we use standard notation.

First note that

$$\bar{g}^{t_{r_1 \oplus r_2}} = (r_1 \oplus r_2)\bar{g} = (r_1\bar{g})(r_2\bar{g}) = \bar{g}^{t_{r_1}} \bar{g}^{t_{r_2}} = \bar{g}^{t_{r_1} + t_{r_2}}$$

for all $g \in G$. Since $G/Z(G)$ is a free abelian group, $\phi(r_1 \oplus r_2) = t_{r_1 \oplus r_2} = t_{r_1} + t_{r_2} = \phi(r_1) + \phi(r_2)$.

Similarly with the multiplication operation,

$$\bar{g}^{t_{r_1 \odot r_2}} = (r_1 \odot r_2)\bar{g} = r_1(r_2(\bar{g})) = r_1(\bar{g}^{t_{r_2}}) = \bar{g}^{t_{r_1} t_{r_2}}$$

for any $g \in G$. Again, since $G/Z(G)$ is torsion-free, $\phi(r_1 \odot r_2) = t_{r_1 \odot r_2} = t_{r_1} t_{r_2} = \phi(r_1)\phi(r_2)$. Notice also that $\bar{g}^{t_{1_R}} = 1_R \bar{g} = \bar{g}$ for all g , because since R is (identified with) a subring of $\text{End}(G/Z(G))$, the identity element of R is the identity endomorphism of $G/Z(G)$. This implies that $\phi(1_R) = t_{1_R} = 1$, completing the proof that ϕ is a ring homomorphism.

Moreover, ϕ is exhaustive, because given $k \in \mathbb{Z}$, we have

$$\phi\left(\sum_{i=1}^k 1_R\right) = \sum_{i=1}^k \phi(1_R) = k.$$

Finally, notice that if $\phi(r) = t_r = 0$ for some r , then $r\bar{g} = \bar{1}$ for all $g \in G$. Hence, r is the 0 element of R (which is the null endomorphism of $\text{End}(G/Z(G))$), i.e. $r = 0_R$. We conclude that ϕ is a ring isomorphism. This proves that any ring of scalars of G is isomorphic to \mathbb{Z} . In particular, this is true of the largest ring of scalars of G . \square

We remark that Theorem 3.5 is still true as long as the complement of the commutativity graph between the a_i 's is connected.

Proposition 3.6. *Suppose \mathbb{Z} is the largest ring of scalars of a τ_2 -group G . Then G cannot be decomposed into a direct product of non-abelian subgroups.*

Proof. Suppose to the contrary that $G = H \times K$ for some non-abelian subgroups H, K of G . Then H and K are non-abelian τ_2 -groups. It is immediate to verify that $Z(G) = Z(H) \times Z(K)$, and that $G/Z(G) \cong H/Z(H) \times K/Z(K)$. Moreover, using that

$$[(h_1, k_1), (h_2, k_2)] = (h_1^{-1}, k_1^{-1})(h_2^{-1}, k_2^{-1})(h_1, k_1)(h_2, k_2) = ([h_1, h_2], [k_1, k_2]) \quad (3)$$

for all $h_1, h_2 \in H, k_1, k_2 \in K$, we obtain $G' = H' \times K'$.

Consider the natural ring actions of \mathbb{Z}^2 on $H/Z(H) \times K/Z(K)$ and on $H' \times K'$ defined by component-wise exponentiation (or component-wise multiplication if one is using additive notation):

$$(r_1, r_2)(h, k) = (h^{r_1}, k^{r_2}), \quad (4)$$

for $(r_1, r_2) \in \mathbb{Z}^2$ and (h, k) in $H/Z(H) \times K/Z(K)$ or in $H' \times K'$. Note that, by fixing a tuple (r_1, r_2) in (4) we obtain an endomorphism of $H/Z(H) \times K/Z(K)$ (or $H' \times K'$).

We next show that these actions are faithful. Suppose

$$(r_1, r_2)(hZ(H), kZ(K)) = (Z(H), Z(K))$$

for all $h, k \in H, K$. Then $h^{r_1} \in Z(H)$ and $k^{r_2} \in Z(K)$ for all $h, k \in H, K$. Since both H and K are τ_2 -groups, $H/Z(H)$ and $K/Z(K)$ are free abelian, and so either $H = Z(H)$ or $r_1 = 0$, and similarly for K and r_2 . Since H and K are non-abelian, $r_1 = r_2 = 0$. This shows that the action (4) is faithful, which implies there exists an embedding of rings $\mathbb{Z}^2 \hookrightarrow \text{End}(H/Z(H) \times K/Z(K))$. Similar arguments show that \mathbb{Z}^2 also embeds (as a ring) into $\text{End}(H' \times K')$.

Moreover, using (3),

$$\begin{aligned} [(r_1, r_2)(u_1, u_2), (v_1, v_2)] &= [(u_1^{r_1}, u_2^{r_2}), (v_1, v_2)] = ([u_1^{r_1}, v_1], [u_2^{r_2}, v_2]) = \\ &= ([u_1, v_1]^{r_1}, [u_2, v_2]^{r_2}) = [(u_1, u_2), (r_1, r_2)(v_1, v_2)]. \end{aligned}$$

For all $(r_1, r_2) \in \mathbb{Z}^2$ and all $(u_1, u_2), (v_1, v_2) \in H/Z(H) \times K/Z(K)$. Similarly:

$$\begin{aligned} [(r_1, r_2)(u_1, u_2), (v_1, v_2)] &= ([u_1^{r_1}, v_1], [u_2^{r_2}, v_2]) = \\ &= ([u_1, v_1]^{r_1}, [u_2, v_2]^{r_2}) = (r_1, r_2)([u_1, v_1], [u_2, v_2]) = (r_1, r_2)[(u_1, u_2), (v_1, v_2)]. \end{aligned}$$

Hence $[\mathbf{r}\mathbf{u}, \mathbf{v}] = [\mathbf{u}, \mathbf{r}\mathbf{v}] = \mathbf{r}[\mathbf{u}, \mathbf{v}]$ for all $\mathbf{r} \in \mathbb{Z}^2$ and all $\mathbf{u}, \mathbf{v} \in H/Z(H) \times K/Z(K)$. Thus, \mathbb{Z}^2 is a ring of scalars of $G = H \times K$. By definition, \mathbb{Z}^2 embeds into the largest ring of scalars of G , which is \mathbb{Z} by hypothesis - a contradiction. \square

3.3 C-small elements and the rank of $M(R)$

Through this section we let G denote a nilpotent group of the form $G = N/\langle\langle R \rangle\rangle$, where N is the s -step rank- m free nilpotent group with basic generating set $A = \{a_1, \dots, a_m\}$, and $R = \{g_1, \dots, g_r\}$ is a set of r words on the alphabet $A^{\pm 1}$ (projected onto N). We assume $s \geq 2$, otherwise G is abelian. Recall that $M(R)$ denotes the $r \times m$ matrix whose (i, j) -th entry is the sum of the exponents of the a_j 's appearing in g_i . Remember also that G_i denotes the i -th term in the lower central series of G , and that $Is(G_i)$, the *isolator* of G_i , is defined as $Is(G_i) = \{g \in G \mid g^n \in G_i \text{ for some } n \in \mathbb{Z}, n \neq 0\}$. We say that G is *regular* if $Z(G) \leq Is(G')$.

We also adopt the following standard convention regarding the projection of elements onto factor groups: Suppose K has been obtained from another group H by adding some relations, and let $\pi : H \rightarrow K$ be the natural projection of H onto K . To avoid constantly referring to π , we will speak of *elements h from H seen in K* (or *projected onto K*), rather than of elements $\pi(h)$. Similarly, for $h_1, h_2 \in H$, instead of writing $\pi(h_1) = \pi(h_2)$, we will say that $h_1 = h_2$ in K .

The goal of this section is to prove Theorem 3.7, which we re-state here.

Theorem 3.7. *Suppose that $M(R)$ has full rank, i.e. $\text{rank}(M(R)) = \min\{r, m\}$. Then the following holds:*

1. *If $r \leq m - 2$, then G is regular, G/G_3 is virtually free nilpotent of rank $m - r$, and G/G_3 does not decompose as the direct product of non-virtually abelian groups.*
2. *If $r = m - 1$, then G is virtually abelian.*
3. *If $r \geq m$, then G is finite.*
4. *If $r \geq m + 1$, then G is finite and abelian.*

We remark that the statement concerning abelianity in Item 4 is proved by Cordes, Duchin, Duong, Ho, and Sánchez in [1].

In this section we also prove the following result regarding the Diophantine problem over nilpotent groups:

Theorem 3.8. *Suppose that $M(R)$ has full rank. Then the following holds:*

1. *If $r \leq m - 2$, then \mathbb{Z} is e -interpretable in G and the Diophantine problem over G is undecidable.*
2. *If $r \geq m - 1$ then Diophantine problem over G is decidable.*

We start by showing that $M(R)$ may be assumed to have a very simple form:

Lemma 3.9. *By performing Nielsen transformations on A and R , we may assume without loss of generality that $M(R)$ is in Smith normal form. In this case, for $i = 1, \dots, \min\{r, m\}$, we have $g_i = a_i^{\alpha_i} c_i$ in N , for some nonzero integers α_i and some $c_i \in N'$. Also, if $m < r$, then $g_i \in N'$ for all $i = m + 1, \dots, r$.*

Proof. For convenience, during this proof we denote N and $M(R)$ by $N(A)$ and $M(A, R)$. First suppose that A' has been obtained from A by performing a Nielsen transformation $a_j \mapsto a'_j = a_i^{\pm 1} a_j$. Then A' is a basic generating set of N , and $N(A)/\langle\langle R \rangle\rangle = N(A')/\langle\langle R' \rangle\rangle = G$, where R' denotes the set of words from R rewritten as words on the alphabet A' (by replacing each a_j with $a_i^{\mp 1} a'_j$). A similar situation holds when performing a Nielsen transformation on R (in this case, the set A does not require any rewriting). Hence, we may apply as many Nielsen transformations to A and R as we wish, rewriting the relators R accordingly when needed.

Recall that the Smith normal form of $M(A, R)$ can be obtained by successively adding or subtracting a row of $M(A, R)$ to another different row, or a column to another different column, and by reordering rows and columns. Of course, this does not change the rank of the matrix. We are now going to prove that, for each matrix M' obtained from $M(A, R)$ by applying one of these operations, one may perform Nielsen transformations to A or R (and then rewrite R if needed) so that, for the resulting sets A', R' , the matrix $M(A', R')$ is precisely M' . The lemma follows from this claim and from the observations made in the previous paragraph.

To prove the claim suppose first that R' has been obtained from R by applying a Nielsen transformation to it, say

$$R' = \{g_1, \dots, g_{j-1}, g'_j, g_{j+1}, \dots, g_r\}, \quad \text{where } g'_j = g_i^{\pm 1} g_j, \quad (i \neq j).$$

Then $ab(g'_j) = ab(g_i) \pm ab(g_j)$, where $ab : N \rightarrow N_{ab}$ is the natural projection of N onto its abelianization. It follows that the matrix $M(A, R')$ is precisely $M(A, R)$ after adding (subtracting) the i -th row to the j -th row of $M(A, R)$. Similarly, suppose A' has been obtained by means of a Nielsen move:

$$A' = \{a_1, \dots, a_{j-1}, a'_j, a_{j+1}, \dots, a_m\}, \quad \text{where } a'_j = a_i^{\pm 1} a_j, \quad (i \neq j),$$

and let R' be the words R written on the alphabet A' by replacing each a_j with $a_i^{\mp 1} a'_j$. Suppose $ab(g_k)$ has coordinates $(\lambda_{k1}, \dots, \lambda_{km})$ with respect to the basis $ab(A) = (ab(a_1), \dots, ab(a_m))$. Then the coordinates of $ab(g_k)$ with respect to $ab(A')$ are the same, except for the i -th coordinate, which is now $\lambda_{ki} \mp \lambda_{kj}$. Therefore, $M(A', R')$ is precisely $M(A, R)$ after subtracting (adding) the j -th column to the i -th column of $M(A, R)$. Finally, notice that interchanging two relators r_i, r_j (or two generators a_i, a_j) results in switching the i -th and j -th rows (columns) of $M(A, R)$. Hence, any row or column operation needed to obtain the Smith normal form of $M(A, R)$ can be ‘realized’ as a Nielsen transformation on A and R (and, maybe, rewriting R accordingly). This completes the proof of the claim.

In conclusion: by successively applying Nielsen transformations to A and R (and rewriting the words from R accordingly), one obtains new sets, A_0 and R_0 , such that A_0 is a basic generating set of N , $N(A_0)/\langle\langle R_0 \rangle\rangle = G$, and $M(A_0, R_0)$ is in Smith normal form. \square

In views of the previous Lemma 3.9, we assume for the rest of the section that $g_i = a_i^{\alpha_i} c_i$ in N for all $i = 1, \dots, \min\{r, m\}$, for some *nonzero* integers

α_i and some elements $c_i \in N'$. This allows us to prove the last three items of Theorem 3.7:

Theorem 3.10 (Items 2, 3, and 4 of Theorem 3.7). *If $\text{rank}(M(R)) = \min\{r, m\}$, then:*

2. *If $r = m - 1$, then G is virtually abelian.*

3. *If $r \geq m$, then G is finite.*

4. *If $r \geq m + 1$, then G is finite and abelian.*

Proof. Suppose first that $r \geq m$. Then $g_i = a_i^{\alpha_i} c_i$ in N for all $i = 1, \dots, m$, with $\alpha_i \in \mathbb{Z} \setminus \{0\}$ and $c_i \in N'$ for all i . Hence, in $G = N / \langle\langle R \rangle\rangle$, we have $a_i^{\alpha_i} = c_i^{-1} \in G'$, which implies that each a_i has finite order in G/G' . Since G/G' is abelian and it is generated by the a_i 's, G/G' is finite of order, say, k . In [14] it is proved that there is an epimorphism

$$\phi_i : G/G' \otimes G_{i-1}/G_i \rightarrow G_i/G_{i+1}, \quad (5)$$

for all $i \geq 2$, where \otimes denotes the tensor product of abelian groups. Using additive notation for abelian groups, we obtain

$$k(G'/G_3) = k\phi_2(G/G' \otimes G/G') = \phi_2(G/G' \otimes kG/G') = 1,$$

It follows that G'/G_3 is finite. We can now use induction and (5) to show that G_i/G_{i+1} is finite for all $i > 1$. By construction, G is a quotient of a free s -step free nilpotent group, so $G_{s+1} = 1$, and G_s is finite. This makes G_{s-1} a finite-by-finite group, which implies that G_{s-1} is finite. Again by induction, we see that G is finite as well.

The proof of Items 3 and 4 follows from the above and from [1], where, as mentioned previously, its authors show that if $r \geq m + 1$, then G is abelian a.a.s.

We proceed to prove Item 2. Suppose that $r = m - 1$. Similarly as before, $g_i = a_i^{\alpha_i} c_i$ in N for all $i = 1, \dots, m - 1$ ($\alpha_i \neq 0$), and $a_i^{\alpha_i} \in G'$. Let $\alpha = \alpha_1 \dots \alpha_{m-1} \neq 0$. Then

$$[a_i, a_j]^{\alpha} G_3 = [a_i^{\alpha}, a_j] G_3 = G_3 \quad \text{for all } 1 \leq i < j \leq m.$$

Since the set $\{[a_i, a_j] \mid i < j\}$ generates the abelian group G'/G_3 , we obtain that G'/G_3 is finite. By the same argument as above, G' is finite as well.

We now prove that G is virtually abelian. We proceed by induction on s . The case $s = 1$ is clear. For the case $s > 1$, observe that $\bar{G} = G/G_s$ is nilpotent of class $s - 1$, and that $(\bar{G})'$ is finite (because G' is). By induction, \bar{G} is virtually abelian, and hence it has an abelian subgroup \bar{H} of finite index. Let H be the full preimage of \bar{H} in G . By the third homomorphism theorem, H has finite index in G . Moreover, $H'/G_s = (H/G_s)' = \bar{H}' = 1$, and thus $H' \leq G_s$. Now, from $[H, H'] \leq [H, G_s] \leq [G, G_s] \leq G_{s+1} = 1$, we obtain that H is 2-step nilpotent. Then, since H is 2-nilpotent,

$$[h_1, h_2^t] = [h_1, h_2]^t = 1$$

for any two $h_1, h_2 \in H$, and so $H^t = \{h^t \mid h \in H\}$ belongs to the center of H . Consequently, $\langle H^t \rangle$ is an abelian normal subgroup of H , and $H/\langle H^t \rangle$ is a finitely generated nilpotent group of finite exponent t , which implies that $H/\langle H^t \rangle$ is finite (see [14], Corollary 9 in Chapter 1). We conclude that $\langle H^t \rangle$ is a finite-index abelian subgroup of H and so that H is virtually abelian. Since H has finite index in G , so is G . \square

For the next results we focus on the case $s = 2$, i.e. we assume N is 2-step nilpotent. In this case, G is 2-step nilpotent as well. In particular, $[\cdot, \cdot] : G \times G \rightarrow G'$ behaves ‘bilinearly’, and $G' \leq Z(G)$. The same occurs in N . We will use these facts extensively during the rest of the section, often without explicitly referring to them.

Lemma 3.11. *Suppose $s = 2$. Then the projection of a_i onto $G/Is(G_3)$ belongs to the center of $G/Is(G_3)$, for all $i = 1, \dots, \min\{r, m\}$.*

Proof. Since $g_i = 1$ in G , we have $a_i^{\alpha_i} = c_i^{-1} \in G' \leq Z(G)$ for all $i = 1, \dots, \min\{r, m\}$. Hence, in G : $[a_i, g]^{\alpha_i} = [a_i^{\alpha_i}, g] = 1$ for all $g \in G$. Since $\alpha_i \neq 0$, the commutator $[a_i, g]$ belongs to $Is(G_3)$, and thus $[a_i, g] = 1$ in $G/Is(G_3)$ for all g and all $i = 1, \dots, \min\{r, m\}$, as needed. \square

We next provide the main two technical results of this section.

Lemma 3.12. *Suppose $s = 2$ and $r \leq m$, and let h be an element from N of the form*

$$h = a_{r+1}^{\gamma_{r+1}} \dots a_m^{\gamma_m} c,$$

with $c \in N'$ and $\gamma_i \in \mathbb{Z}$ for all i . Assume $h = 1$ in G . Then $\gamma_{r+1} = \dots = \gamma_m = 0$, and $c \in \langle \{[a_i, a_j] \mid i \leq r \text{ or } j \leq r\} \rangle$.

Proof. Since $h = 1$ in G , h can be written in N as a product of conjugates of elements from R and their inverses. Hence, there exists a sequence of elements $w_j \in N$, signs $\epsilon_j \in \{+1, -1\}$, and relators

$$g_{i_j} = a_{i_j}^{\alpha_{i_j}} c_{i_j} \in R = \{g_1, \dots, g_r\}, \quad (j = 1, \dots, p),$$

such that, in N ,

$$h = a_{r+1}^{\gamma_{r+1}} \dots a_m^{\gamma_m} c = \prod_{j=1}^p w_j^{-1} a_{i_j}^{\epsilon_j \alpha_{i_j}} c_{i_j}^{\epsilon_j} w_j \quad (6)$$

We now wish to write the Malcev representation of the right hand-side of (6). To do so, it suffices to move the a_{i_j} ’s to the left by repeatedly applying the identity: $a_s a_t = a_t a_s [a_s, a_t]$, which holds for all s, t . Once this is done, we move all commutators introduced this way, and all the c_{i_j} ’s (which belong to the center of N) to the left of (6). The elements w_j^{-1} and w_j then cancel between themselves. Notice that during these operations we only introduce commutators of the form $[a_s, a_t]$ with $s \leq r$ or $t \leq r$ (because $i_j \in \{1, \dots, r\}$ for all j). Denote

by C_0 the set of all such commutators. For $k = 1, \dots, r$, let λ_k be the sum of those ϵ_i 's for which $i_j = k$, i.e.

$$\lambda_k = \sum_{i_j=k} \epsilon_j.$$

Then (6) can be written as

$$h = a_{r+1}^{\gamma_{r+1}} \dots a_m^{\gamma_m} c = \left(\prod_{k=1}^r a_k^{\lambda_k \alpha_k} \prod_{k=1}^r c_k^{\lambda_k} \right) d, \quad (7)$$

where d is a product of commutators from C_0 . This equality takes place in the free 2-step nilpotent group N . By uniqueness of Malcev coordinates (see (1)), $\lambda_k = 0$ for all $k \leq r$, and $\gamma_k = 0$ for all $k \geq r+1$. It follows that, in N , $h = c = d \in \langle C_0 \rangle$, as needed. \square

The reason why we need $M(R)$ to have full rank r is that, otherwise, $\alpha_k = 0$ for some k 's. Then in the proof above, (7) does not imply $\lambda_k = 0$ for all k . As a consequence, some $c_k^{\lambda_k}$ may be non-trivial, allowing commutators not belonging to C_0 to be on the right hand-side of (7).

Corollary 3.13. *Assume $s = 2$ and $r \leq m - 1$. Let $h \in N$ be such that $[a_\ell, h]^n = 1$ in G , for some $\ell \in \{r+1, \dots, m\}$ and $n \in \mathbb{Z} \setminus \{0\}$. Then in N ,*

$$h = a_1^{\gamma_1} \dots a_r^{\gamma_r} a_\ell^{\gamma_\ell} c$$

for some $c \in N'$ and $\gamma_i \in \mathbb{Z}$, $i = 1, \dots, r, \ell$.

Proof. There exists a unique $c \in N'$ and unique integers $\gamma_i \in \mathbb{Z}$ such that, in N , $h = \prod_{k=1}^m a_k^{\gamma_k} c$. Using that $[\cdot, \cdot]$ behaves 'bilinearly' in 2-step nilpotent groups, and that $N' \subseteq Z(N)$, we obtain

$$[a_\ell, h]^n = [a_\ell, \prod_{k=1}^m a_k^{\gamma_k} c]^n = \prod_{k=1}^m [a_\ell, a_k]^{n\gamma_k}. \quad (8)$$

Since $[a_\ell, h]^n = 1$ in G , (8) and Lemma 3.12 ensure that, in N ,

$$\prod_{k=1}^m [a_\ell, a_k]^{n\gamma_k} \in \langle \{[a_i, a_j] \mid i \leq r \text{ or } j \leq r\} \rangle.$$

By unicity of Malcev coordinates, and because $\ell \geq r+1$, we obtain $\gamma_k = 0$ for all $k \geq r+1$, $k \neq \ell$. The result follows. \square

Theorem 3.14 (First part of Item 1 from Theorem 3.7). *If $r \leq m - 2$ and $M(R)$ has rank r , then G is regular, and G/G_3 is virtually free nilpotent of rank $m - r$.*

Proof. We start by showing that G is regular. First note that if G/G_3 is regular, then G is regular as well. Indeed, let $x \in N$ be such that, in G , $x \in Z(G)$. Then, in G/G_3 , $x \in Z(G/G_3)$. Assuming G/G_3 is regular, $x^k \in (G/G_3)'$ for some $k \neq 0$. This implies that, in G , x^k is a product of commutators times an element from G_3 . Since $G_3 \leq G'$, we obtain $x \in Is(G')$, as needed.

Thus, it suffices to show that G/G_3 is regular. By the same reasons as in Part 1, we assume without loss of generality that $s = 2$, in which case $G_3 = 1$, and $G/G_3 = G$ is 2-step nilpotent. As before, we denote $T = Is(G_3)$ and $G_0 = G/T$.

Let x be an element from N that belongs to $Z(G)$ in G , and write $x = a_1^{\gamma_1} \dots a_m^{\gamma_m} c$ in N , for some $\gamma_i \in \mathbb{Z}$ and $c \in N'$. Recall that the relators from R have the form $a_i^{\alpha_i} c_i$, for some $\alpha_i \in \mathbb{Z} \setminus \{0\}$, $c_i \in N'$, $i = 1, \dots, r$. In particular, $a_i^{\alpha_i} \in G' \leq Z(G)$ in G . Denote $\alpha = \alpha_1 \dots \alpha_r \neq 0$. Using the identities $a_i a_j = a_j a_i [a_i, a_j]$, we obtain in N :

$$x^\alpha = \prod_i a_i^{\alpha \gamma_i} d,$$

for some $d \in N'$. The factor $a_1^{\alpha \gamma_1} \dots a_r^{\alpha \gamma_r}$ projects into $G' \leq Z(G)$, and x^α projects into $Z(G)$, because x does. It follows that the element

$$y = a_{r+1}^{\alpha \gamma_{r+1}} \dots a_m^{\alpha \gamma_m},$$

belongs to $Z(G)$ in G . Consequently, $[a_\ell, y] = 1$ in G for all $\ell = 1, \dots, m$. Since $r \leq m - 2$, using unicity of Malcev coordinates in N and Corollary 3.13 *twice*, we obtain $\alpha \gamma_i = 0$ for all $i = r + 1, \dots, m$. But then

$$x^\alpha = a_1^{\alpha \gamma_1} \dots a_r^{\alpha \gamma_r} d \in G',$$

because $a_i^{\alpha_i} \in G'$ for all $i = 1, \dots, r$. This shows that G is regular.

We now prove that G/G_3 is virtually free nilpotent of rank $m - r$. Observe that

$$G/G_3 = \langle A \mid [A, A, A] = 1, R = 1 \rangle = \langle A \mid [A, A, A] = 1 \rangle / \langle\langle R \rangle\rangle = \tilde{N} / \langle\langle R \rangle\rangle,$$

where \tilde{N} is the 2-step rank- m free nilpotent group with basic generating set A . Hence, for fixed A and R , G/G_3 is independent of the nilpotency step s of N ($s \geq 2$). Thus, we can assume without loss of generality that $s = 2$. In this case, $Is(G_3) = Is(1)$ is the torsion subgroup of G , and $G/G_3 = G$ is 2-step nilpotent. For convenience we denote $T = Is(G_3)$ and $G_0 = G/T$.

Let H_0 be the subgroup of G_0 generated by $A_{>r} = \{a_{r+1}, \dots, a_m\}$, and let $N_{>r}$ be the 2-step free nilpotent group with basic generating set $A_{>r}$. Denote by $\theta : N_{>r} \rightarrow H_0$ the natural map from $N_{>r}$ into H_0 . Clearly, θ is an epimorphism. We claim that it is one-to-one. To prove so, assume to the contrary that $\ker(\theta) \neq 1$. Then $\ker(\theta) \cap Z(N_{>r}) \neq 1$, because if the intersection was trivial, then no element $g \in \ker(\theta)$ would commute with all $N_{>r}$, and so $[\ker(\theta), N_{>r}] \neq \{1\}$, but also $[\ker(\theta), N_{>r}] \leq \ker(\theta) \cap Z(N_{>r})$, a contradiction. Let x be a nontrivial element from $\ker(\theta) \cap Z(N_{>r})$. Since $N_{>r}$ is a free nilpotent group,

$Z(N_{>r}) = N'_{>r}$, and hence x is a non-trivial product of commutators belonging to $[A_{>r}, A_{>r}]$. From $x \in \ker(\theta)$ we obtain that such product of commutators equals 1 in G_0 , and thus that a power of them equals 1 in G . This contradicts Lemma 3.12. Consequently, $\ker(\theta) = 1$, and θ is an isomorphism.

We now prove that H_0 has finite index in G_0 . Indeed, by Lemma 3.11, $a_i \in Z(G_0)$ for all $i = 1, \dots, r$. Therefore, $G'_0 \leq \langle [A_{>r}, A_{>r}] \rangle = H'_0$. Also, by construction, $a_i^{\alpha_i} \in G'_0 \leq H_0$ for all $i = 1, \dots, r$. One can immediately verify that H_0 is normal in G_0 (because a_1, \dots, a_r are central), and that G_0/H_0 is an abelian group generated by (the projection of) a_1, \dots, a_r . Finally, since $\alpha_i \neq 0$ for all i , each a_i has order at most α_i in G_0/H_0 . It follows that $|G_0/H_0| < \infty$, as required.

Let now $H = \langle A_{>r} \rangle$ be the subgroup of G generated by $A_{>r}$, and denote by $\pi : G \rightarrow G_0 = G/T$ the natural projection of G onto G_0 . Clearly, $\pi(H) = H_0$, with $\pi(a_i) = a_i$. Since H_0 is a free 2-step nilpotent group generated by $A_{>r}$, and H is a 2-step nilpotent group generated by $A_{>r}$, there is an inverse map π^{-1} , so π is an isomorphism. Finally, the index $|G : H|$ is finite because $|G : H| = |G : HT| |HT : H| \leq |G_0 : H_0| |T|$, and because the torsion subgroup of a finitely generated nilpotent group is normal and finite [14]. \square

We leave the statement of Theorem 3.7 about direct indecomposability of G/G_3 for the end of the section. We next prove one of our main results about the Diophantine problem over nilpotent groups.

Theorem 3.8. *Let $G = N_{s,m}/\langle\langle R \rangle\rangle$, where $N_{s,m} = N_{s,m}(A)$ is a free nilpotent group of nilpotency class $s \geq 2$ and rank m with basis A and R is a set of r words of length ℓ in the alphabet $A^{\pm 1}$. Then the following hold asymptotically almost surely as $\ell \rightarrow \infty$:*

1. *If $r \leq m - 2$, then \mathbb{Z} is e-interpretable in G and the Diophantine problem over G is undecidable.*
2. *If $r \geq m - 1$ then Diophantine problem over G is decidable.*

Proof. Item 2 follows from Theorem 3.7. Assume therefore that $r \leq m - 2$. We are going to show that \mathbb{Z} is e-definable in $G/Is(G_3)$, in which case Item 1 follows immediately. Since $G/Is(G_3)$ is a quotient of G/G_3 , and for the same reasons as when proving that G/G_3 is virtually free nilpotent in Theorem 3.14, we assume that $s = 2$. Then $G_3 = 1$ and G is 2-step nilpotent. Again, we use the notation $T = Is(G_3) = Is(1)$ and $G_0 = G/T$.

Since G_0 is a f.g. torsion-free 2-step nilpotent group, by Theorem 3.2, it suffices to show that (the projections of) a_{m-1} and a_m are two non-commuting c-small elements of G_0 . With this in mind, suppose that $[a_{m-1}, a_m] = 1$ in G_0 . Then $[a_{m-1}, a_m] \in T$ in G , and thus, by the previous Corollary 3.13, the following identity holds in N :

$$a_m = \left(\prod_{i=1}^r a_i^{\gamma_i} \right) a_{m-1}^{\gamma_{m-1}} c \quad (9)$$

for some $\gamma_i, \gamma_{m-1} \in \mathbb{Z}$, $c \in N'$, $i = 1, \dots, r$ (here we have used the hypothesis $r \leq m - 2$). Since this equality is in N , by uniqueness of Malcev coordinates, $\gamma_{m-1} = 0$ and $\gamma_i = 0$ for all $i = 1, \dots, r$, and thus $a_m \in N'$, a contradiction.

We next show that a_m and a_{m-1} are c-small as elements of G_0 . Suppose that $[a_m, h] = 1$ in G_0 for some $h \in N$. Then $[a_m, h] \in T$ in G , and, again by Corollary 3.13, we have in N :

$$h = \left(\prod_{i=1}^r a_i^{\tau_i} \right) a_m^{\tau_m} d$$

for some $\tau_i, \tau_m \in \mathbb{Z}$, and some $d \in N'$. By Lemma 3.11, the following holds in G_0 :

$$\prod_{i=1}^r a_i^{\tau_i} \in Z(G_0).$$

We conclude that h can be written as $h = a_m^{\tau_m} z$ in G_0 , for $\tau_m \in \mathbb{Z}$ and $z \in Z(G_0)$. From this it follows that a_m is a c-small element of G_0 . The same argument works for a_{m-1} , and, actually, for each a_{r+1}, \dots, a_m .

We have so far proved that \mathbb{Z} is e-definable in $G/Is(G_3)$. It follows that $\mathcal{D}(\mathbb{Z})$ is reducible to $\mathcal{D}(G/Is(G_3))$ by Corollary 2.6. By definition, $G/Is(G_3)$ is G/G_3 modulo its torsion subgroup, and so $\mathcal{D}(G/Is(G_3))$ is reducible to $\mathcal{D}(G/G_3)$, by Proposition 2.8. But $\mathcal{D}(G/G_3)$ is in turn reducible to $\mathcal{D}(G)$ by Proposition 2.7. This completes the proof of the theorem. \square

Theorem 3.15 (Last part of Theorem 3.7). *If $r \leq m - 2$ and $M(R)$ has rank r , then G/G_3 does not decompose as the direct product of two non-virtually abelian groups.*

Proof. By the same reasons as in Theorems 3.14–3.8, it can be assumed without loss of generality that $s = 2$. In this case $G_3 = 1$ and $Is(G_3)$ is the torsion subgroup of G . As usual, denote $G_0 = G/Is(G_3)$. However, this time denote $Is(G_3)$ by $T(G)$ instead of simply by T .

As we saw in the proof of Theorem 3.8, a_i is c-small in G_0 for all $i = r + 1, \dots, m$. Hence, by Proposition 3.6, G_0 is indecomposable as the direct product of non-abelian groups.

Now assume that $G = H \times K$ for some groups H and K . Then $T(G) = T(H) \times T(K)$, and $G_0 = G/T(G) \cong H/T(H) \times K/T(K)$. We have seen that G_0 is indecomposable as the direct product of non-abelian groups. Hence, either $H/T(H)$ or $K/T(K)$ is abelian. Assume this is the case for $H/T(H)$. We next show that H is virtually abelian. Indeed, since $H/T(H)$ is abelian, $[h_1, h_2]$ is trivial in $H/T(H)$ for all $h_1, h_2 \in H$. Therefore, $H' \leq T(H)$. On the other hand, H is a finitely generated nilpotent group, and therefore $T(H)$ is finite (see [14]). This implies that H' is finite as well. By Lemma 2.1, H is virtually abelian. \square

4 Linear independence of random walks

Recall that, given some words $R = \{g_1, \dots, g_r\}$ on $A^{\pm 1} = \{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$, we let $M(R) = M(\{g_1, \dots, g_r\})$ denote the $r \times m$ matrix whose (i, j) -th entry is the sum of the exponents of all the a_j 's appearing in g_i .

In this section we prove:

Theorem 4.1. *Let R be a set of r words, each one chosen randomly among all words of length ℓ on the alphabet $A^{\pm 1} = \{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$. Then $M(R)$ has full rank (i.e. $\text{rank}(M(R)) = \min\{r, m\}$) asymptotically almost surely as $\ell \rightarrow \infty$.*

A random word of length ℓ on $A^{\pm 1}$ may also be seen as a random walk of ℓ steps in the free abelian group \mathbb{Z}^m with basis A . In this language, Theorem 4.1 may be rephrased as *the final positions of $r \leq m$ independent random walks in \mathbb{Z}^m are linearly independent a.a.s., seen as vectors of \mathbb{R}^m .*

We next provide some basics regarding such walks, referring the reader to [7] for a comprehensive treatment of the subject. We will use the letter n , rather than ℓ , to denote the number of steps of a random walk.

4.1 Central Limit Theorems

One may model a *random walk* in \mathbb{Z}^m (with respect to the basis A) in the following way: Let $(X_t \mid t \geq 1)$ be an infinite sequence of independent random variables taking values in $A^{\pm 1}$, with a uniform probability of $1/2m$. The random variable $S_n = X_1 + \dots + X_n$ is called a *random walk in \mathbb{Z}^m of n steps*, and the infinite sequence $(S_0, S_1, \dots, S_n, \dots)$, $S_0 = 0$, a *random walk in \mathbb{Z}^m* .

Each coordinate $x_{t,i}$ of $X_t = (x_{t,1}, \dots, x_{t,m})$ (with respect to the basis A) is a random variable taking values 1 and -1 with probability $1/2m$, and 0 with probability $1 - 1/m$. Moreover, since the X_t 's are independent, so are the $x_{t,i}$'s, for a fixed i . We now use the Central Limit Theorem (CLT) to find the asymptotic behavior (as $n \rightarrow \infty$, with i fixed) of the coordinates $s_{n,i} = x_{1,i} + \dots + x_{n,i}$ of $S_n = (s_{n,1}, \dots, s_{n,m})$. Observe that each $x_{t,i}$ has expected value and variance

$$E(x_{t,i}) = 1 \frac{1}{2m} + (-1) \frac{1}{2m} = 0, \quad \text{Var}(x_{t,i}) = E(x_{t,i}^2) - E(x_{t,i})^2 = \frac{2}{2m} = \frac{1}{m},$$

and hence, by the CLT, $s_{n,i}/\sqrt{n}$ converges to the normal distribution $N(0, 1/m)$ with expectation 0 and standard deviation $1/\sqrt{m}$. More precisely, for every $i = 1, \dots, m$, and $M, N \in \mathbb{R} \cup \{\pm\infty\}$,

$$\lim_{n \rightarrow \infty} \left[\mathbb{P} \left(N < \frac{s_{n,i}}{\sqrt{n}} < M \right) \right] = \mathbb{P}(N < \xi < M), \quad (10)$$

where ξ is a variable with distribution $N(0, 1/m)$. Moreover, the $x_{t,i}$'s have third moment $E(|x_{t,i}|^3) = 1 < \infty$, and thus, by the Berry-Esseen Theorem, the

convergence in (10) is uniform on M, N , i.e. there exists a constant C such that, for all i, N, M and n ,

$$\left| \mathbb{P} \left(N < \frac{s_{n,i}}{\sqrt{n}} < M \right) - \mathbb{P} (N < \xi < M) \right| \leq \frac{C}{\sqrt{n}}. \quad (11)$$

In particular:

Lemma 4.2. *Given a sequence ϵ_n with $\lim_{n \rightarrow \infty} \epsilon_n = \infty$, the following holds for all i :*

$$\lim_{n \rightarrow \infty} \left[\mathbb{P} \left(\left| \frac{s_{n,i}}{\sqrt{n}} \right| \geq \epsilon_n \right) \right] = 0.$$

Proof. Indeed, it suffices to show that $\mathbb{P}(|s_{n,i}/\sqrt{n}| < \epsilon_n) \rightarrow 1$. First notice that $\mathbb{P}(|\zeta| < \epsilon_n) \rightarrow 1$ for any random variable ζ . Using (11): $\mathbb{P}(|(s_{n,i}/\sqrt{n})| < \epsilon_n) \leq (\mathbb{P}(|\xi| < \epsilon_n) + C/\sqrt{n}) \rightarrow 1$. \square

Intuitively, this result shows that, for large n , the variables $s_{n,i}$ have absolute value not larger than $\epsilon_n \sqrt{n}$ almost surely. For our arguments we will use $\epsilon_n = \ln(n)$, though it suffices to take any sequence that approaches infinity slowly enough.

We will also need the following local version of the Central Limit Theorem for random walks. Given a point $T = (t_1, \dots, t_m) \in \mathbb{Z}^m$, denote by $p_n(T)$ the probability of being at T on the n -th step of a random walk, i.e. $p_n(T) = \mathbb{P}(S_n = T)$. Notice that $p_n(T) = 0$ if the parity of n is different than the parity of $t_1 + \dots + t_m$. Hence the distribution of $p_n(T)$ is actually represented by $p_n(T) + p_{n+1}(T)$ (as one of them is zero).

Local Central Limit Theorem [7]. *Following the notation above, there exists a constant c_0 such that, for all $T \in \mathbb{Z}^m$ and $n \in \mathbb{N}$,*

$$\left| p_n(T) + p_{n+1}(T) - \tilde{p}_n(T) \right| < \frac{c_0}{n^{(m+2)/2}},$$

where

$$\tilde{p}_n(T) = \frac{1}{(2\pi n)^{m/2} c_1} e^{-\frac{J(T)^2}{2n}},$$

where c_1 is a positive constant and $J(\cdot)^2$ is a positive definite quadratic form.

This provides an upper bound for $p_n(T)$:

Proposition 4.3. *There exists a constant c_3 such that, for all $T \in \mathbb{Z}^m$ and $n \in \mathbb{N}$,*

$$p_n(T) \leq \frac{c_3}{n^{m/2}}.$$

Proof. Using Theorem 4.1,

$$p_n(T) \leq |p_n(T) + p_{n+1}(T)| \leq \tilde{p}_n(T) + \frac{c_0}{n^{(m+2)/2}} \leq \frac{c_2}{n^{m/2}} + \frac{c_0}{n^{(m+2)/2}} \leq \frac{c_3}{n^{m/2}},$$

for some positive constants c_2, c_3 . \square

4.2 Schwartz-Zippel Lemma and proof of linear independence

Consider r independent random walks of n steps in \mathbb{Z}^m : $S_{j,n} = (s_{j,n,1}, \dots, s_{j,n,m})$, $j = 1, \dots, r$ (we maintain the notation of the previous section, adding an extra subindex j when appropriate), and let M_n be the $r \times m$ matrix whose j -th row consists in the components $s_{j,n,i}$ of $S_{j,n}$ ($j = 1, \dots, r$). The goal of this subsection is to prove Theorem 4.1, which, in the language of random walks, states that

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{rank}(M_n) = \min(r, m)] = 1. \quad (12)$$

In other words, that *the matrices M_n have full rank asymptotically almost surely as n tends to infinity*.

Given an $r \times m$ matrix $M = (m_{i,j} \mid i, j)$ with integer entries, let $f(M)$ be the polynomial

$$f(M) = f(m_{i,j} \mid i, j) = \sum [\det(M_0)]^2,$$

where the sum runs over all maximal minors M_0 of M . Of course, $f(M) = 0$ if and only if $\det(M_0) = 0$ for all M_0 , i.e. if and only if M does not have full rank. We will need the following combinatorial result to estimate the number of roots of f in bounded sets:

Schwartz-Zippel Lemma. *Let $f(\mathbf{x}) = f(x_1, \dots, x_N) \in \mathbb{C}[x_1, \dots, x_N]$ be a polynomial of degree d on N variables, and let I be a finite set of complex numbers. Then*

$$|\{\mathbf{x} \in I^N \mid f(\mathbf{x}) = 0\}| \leq d|I|^{N-1}.$$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Following the notation above, it is enough to show that $\mathbb{P}(\text{rank}(M_n) = \min(r, m)) \rightarrow 1$ as $n \rightarrow \infty$, or, equivalently, $\mathbb{P}(f(M_n) = 0) \rightarrow 0$. Write $\epsilon_n = \ln(n)$, and let \mathcal{M}_n be the set of $r \times m$ integer matrices M such that $|m_{i,j}| < \sqrt{n}\epsilon_n$ for all entries $m_{i,j}$ of M . Then

$$\mathbb{P}(f(M_n) = 0) \leq \mathbb{P}(f(M_n) = 0, M_n \in \mathcal{M}_n) + \mathbb{P}(M_n \notin \mathcal{M}_n) \quad (13)$$

for all n . Recall that the j -th row of M_n is $S_{j,n} = (s_{j,n,i} \mid i = 1, \dots, m)$. By Lemma 4.2,

$$\begin{aligned} \mathbb{P}(M_n \notin \mathcal{M}_n) &\leq \sum_{j=1}^r \mathbb{P}(S_{j,n} \text{ is such that } |s_{j,n,i}| \geq \sqrt{n}\epsilon_n \text{ for some } i) \\ &\leq \sum_{j,i} \mathbb{P}(S_{j,n} \text{ is such that } |s_{j,n,i}| \geq \sqrt{n}\epsilon_n) \rightarrow 0. \end{aligned}$$

We now bound the first summand of (13). First notice that

$$\mathbb{P}(f(M_n) = 0, M_n \in \mathcal{M}_n) = \sum_{i=1}^{k_n} \mathbb{P}(M_n = T_{n,i}),$$

where $T_{n,1}, \dots, T_{n,k_n}$ are all the zeros of f in \mathcal{M}_n (looking at $r \times m$ matrices as tuples from \mathbb{Z}^{rm}). By the Schwartz-Zippel Lemma 4.2,

$$k_n \leq d (2\sqrt{n}\epsilon_n + 1)^{rm-1} \leq d (3\sqrt{n}\epsilon_n)^{rm-1},$$

where d is the degree of f . Now observe that, for fixed n and i , $M_n = T_{n,i}$ if and only if the j -th row $S_{j,n}$ of M_n is equal to the j -th row $T_{j,n,i}$ of $T_{n,i}$, for all $j = 1, \dots, r$. Using Proposition 4.3 and the fact that the $S_{j,n}$'s form a set of r independent random walks,

$$\mathbb{P}(M_n = T_{n,i}) = \prod_{j=1}^r \mathbb{P}(S_{j,n} = T_{j,n,i}) \leq \left(\frac{c_3}{n^{m/2}} \right)^r$$

for every $i = 1, \dots, k_n$. We conclude that

$$\mathbb{P}(f(M_n) = 0, M_n \in \mathcal{M}_n) \leq d (3\sqrt{n}\epsilon_n)^{rm-1} \left(\frac{c_3}{n^{m/2}} \right)^r = c_4 \frac{\epsilon_n^{rm-1}}{\sqrt{n}},$$

where c_4 is some constant. Since we took $\epsilon_n = \ln(n)$, this last expression tends to 0 as $n \rightarrow \infty$, and hence $\mathbb{P}(f(M_n) = 0) \rightarrow 0$, as needed. \square

5 References

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